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A generalization of the construction of the class operator

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Abstract. It is shown that the construction of the class operator for $SU(2)$ is a partial case of a much more general problem, that of decomposing an operator into components transforming under conjugation according to a given irreducible representation. The problem is solved generally for arbitrary compact groups and some possibilities for extensions of this procedure to the case of non-compact groups are indicated.

1. Introduction

The notion of the class operator, denoting the sum (within the group algebra) of all elements belonging to a particular conjugacy class of group elements, seems to be useful and well established in the realm of the theory of finite groups, cf e.g. Katriel [1]. However, it is only fairly recently that the paper of Fan and Ren [2] has raised some attention to this notion in the Lie case. The authors have posed the problem of evaluating the integral

$$\widetilde{C}(\psi) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \exp[\mathrm{i}\psi(J_x \sin\theta\cos\varphi + J_y \sin\theta\sin\varphi + J_z\cos\theta)]\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \qquad (1)$$

where J_x, J_y and J_z are infinitesimal generators of a representation of the rotation group $SO(3)$ (or $SU(2)$), which, as is not hard to observe, is just the integral over the conjugacy class (within the rotation group $SO(3)$) consisting of all rotations through a fixed angle ψ. The authors calculated the explicit value of the integral in (1) by using some rather sophisticated techniques with a strong quantum field-theoretical flavour. Their result was re-obtained by Backhouse [3], who used simpler techniques of the (conventional) theory of representation of Lie groups, and by Rembieliński [4] who gave an expression for the class operator constructed out of a fixed irreducible representation of $SU(2)$. Moreover, Backhouse outlined a natural extension of the construction to other compact Lie groups.

It is our intention here to show that the construction of the class operator, whether for $SU(2)$ or any other compact Lie group, is just a particular case of a more general construction. Namely, with each function defined on a chosen conjugacy class of a given group G (note that for a Lie group it is in a natural way a smooth manifold) we associate an operator obtained by integrating the given function against conjugation operators of the representation (see (3) and (12) below)—the class operator obtained by choosing the function to be identically 1. To determine an explicit form of such operators one has to

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solve the following problem. For a given operator T on the representation space V of G find a decomposition of it into components which transform under the conjugation according to the irreducible representations of G. The class operator is then the component of the decomposition of $T(g)$, $g \in G$ transforming according to the trivial representation.

This is a natural problem to consider, with roots which can be traced back at least to the work of Wigner on tensor operators. Moreover, the version of the problem we propose lends itself to a nice 'probabilistic' interpretation. Assume that a symmetry operation is performed on a physical system in a random way, with a probability distribution f. For example, imagine a rotation through a fixed angle ψ being performed, however, with an axis of rotation randomly chosen with a probability given by a function $f(\theta, \varphi)$ depending on the orientation of the axis described by its polar coordinates θ, φ. Then the average effect of this operation on the system will be expressed by the integral

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)\exp[\mathrm{i}\psi(J_x\sin\theta\cos\varphi + J_y\sin\theta\sin\varphi + J_z\cos\theta)]\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$$

and on an observable A by an integral of the type (3) with $T(g_0)$ replaced by A. It is perhaps worth pointing out that the resulting average does not belong to the symmetry (e.g. rotation) group any more. (For yet another application of integrals of that kind, see chapter 3 of [7].)

Concerning the general formulation of the problem described above, in this paper we derive an integral representing such components in proposition 1, which is a generalization of the integral (1), and as a corollary we obtain a simple expression for the class operator in the case of an arbitrary compact G. Our method is based on the standard (Peter–Weyl) theory of representation of compact groups. For the case of $SU(2)$ we solve completely the problem of determining irreducible components of $T(g)$ by giving a finite Fourier series expressing them in terms of (modified) Clebsch–Gordan coefficients. As a special case we get the formulas obtained in the above papers.

The integrals we consider here can, in general terms, be written as

$$T(f; H) = \int_G f(x)T(\exp \mathrm{Ad}(x)H)\mathrm{d}x \tag{2}$$

where $f(x)$ is any continuous (or smooth) function on the group G, and Ad denotes the adjoint representation of G on its Lie algebra. However, for the case of non-compact groups like $SU(1, 1)$, such integrals will not, in general, converge unless the function is compactly supported. In the latter case the map $f \mapsto T(f; H)$ is an operator distribution satisfying a simple covariance condition (cf (5) below). These distributions also seem to be a proper starting point for generalizing the construction to the non-compact groups. With this future aim in mind we include here a few remarks concerning such integrals and refer the interested reader to a forthcoming work [6] by the authors on the case of $SU(1, 1)$ for more details.

2. Generalities

If G is a topological group, then by a representation of G on a topological vector space V we shall mean a homomorphism $T: G \to GL(V)$ into the group $GL(V)$ of continuous invertible linear maps on V, which is continuous with respect to the strong operator topology. We shall use a notation (T, V) to denote a representation of any given group on the space V. In the following we shall assume G is a compact group, V a Hilbert space, and denoting by $U(V) \subset GL(V)$ the group of unitary automorphisms of V we assume $T: G \to U(V)$, i.e. (T, V) is a unitary representation (not necessarily irreducible) of G on a Hilbert space V.

If $L(V)$ denotes the space of continuous linear operators on V with the usual Banach norm, then the conjugation $L(V) \ni A \to T(g)AT(g^{-1}) \in L(V)$ defines a continuous representation of G on $L(V)$. If f is a continuous function on G then for an arbitrary fixed $g_0 \in G$ we set

$$T(f; g_0) = \int_G f(x)T(x)T(g_0)T(x^{-1})\mathrm{d}x \tag{3}$$

where the integration is performed with respect to the (bi-invariant, normalized) Haar measure dx on G. Thus, denoting by $\mathcal{C}(G)$ the space of continuous functions on G, we have the mapping

$$\mathcal{C}(G) \ni f \longrightarrow T(f; g_0) \in L(V) \tag{4}$$

defined for any compact G and an arbitrary $g_0 \in G$.

It is a well known fact cf e.g. [7], that the integral (3) converges absolutely. By λ, resp ρ, we shall denote the left, resp right, regular representation of G in $\mathcal{C}(G)$, which is defined as the mapping $f \to \lambda(g)f$, resp $f \to \rho(g)f$, where

$$\lambda(g)f(x) = f(g^{-1}x) \qquad \text{resp} \qquad \rho(g)f(x) = f(xg) \qquad x \in G\,.$$

A straightforward computation gives the following relation:

$$T(g)T(f; g_0)T(g)^{-1} = T(\lambda(g)f; g_0)\,. \tag{5}$$

Assume now that V is a finite-dimensional Hilbert space. The space $L(V)$ of linear operators on V is given an inner product by $(A|B) = \mathrm{Tr}(AB^*)$, which is invariant under the representation of G on $L(V)$ by conjugation, $A \to T(g)AT(g)^{-1}$. We shall denote this representation by $(S, L(V))$, i.e. for any $g \in G$ and $A \in L(V)$ we set

$$S(g)A = T(g)AT(g)^{-1}\,.$$

Then, using general results about representations of compact groups (cf e.g. [8] or [9], sections 6.2 and 6.3 for a statement of the relevant results), one sees that $L(V)$ can be decomposed into irreducible representations in the following way.

Let $\Sigma(V)$ be the set of (classes of) irreducible representations of G which occur in the decomposition of $(S, L(V))$ into irreducibles and for any $\sigma \in \Sigma(V)$ let (T_σ, H_σ) be a fixed representative of this class. Then for each $\sigma \in \Sigma(V)$ there exists a unique subspace $W_\sigma \subset L(V)$, invariant under conjugation and such that in an appropriate basis in W_σ the restriction of $S(g)$ to W_σ is represented by block-diagonal matrices

$$\begin{pmatrix} T_\sigma(g) & 0 & \cdots \\ 0 & T_\sigma(g) & \cdots \\ \cdots & \cdots & \cdots \\ 0 & \cdots & T_\sigma(g) \end{pmatrix} \tag{6}$$

where the number $n(\sigma)$ of blocks along the diagonal is uniquely determined and is called the multiplicity of the class σ in the representation $(S, L(V))$. Then one can write

$$L(V) = \bigoplus_{\sigma\in\Sigma(V)} W_\sigma = \bigoplus_{\sigma\in\Sigma(V)} n(\sigma)H_\sigma\,. \tag{7}$$

Moreover, standard results about matrix coefficients imply the following (cf [9]).

Proposition 1. Let $g_0 \in G$, $g_0 \neq e$, but otherwise arbitrary. Denote by $\mathcal{M}_\sigma$ the subspace of $C(G)$ spanned by the functions $\overline{t_\sigma(g)_{ij}}$, i.e. complex conjugates of matrix elements of the representation (T_σ, H_σ) and by $\mathcal{F}_V \subset C(G)$ the subspace spanned by the conjugate matrix coefficients of representations in $\Sigma(V)$.

(a) The map

$$C(G) \ni f \longrightarrow T(f;\, g_0) \in L(V)$$

maps each space M_σ into the corresponding W_σ and vanishes on the orthogonal complement to $\mathcal{F}_V$ in $C(G)$.

(b) For any $\sigma \in \Sigma(V)$ we set $d_\sigma = \dim H_\sigma$ and let $\chi_\sigma(g) = \mathrm{Tr}(T_\sigma(g))$ denote the character corresponding to the class σ. Then $T(\overline{\chi}_\sigma; g_0) \in W_\sigma$ and

$$T(g_0) = \sum_{\sigma \in \Sigma(V)} d_\sigma T(\overline{\chi}_\sigma; g_0)\,.$$

In other words, if $P_\sigma : L(V) \to W_\sigma$ is the orthogonal projection onto the subspace $W_\sigma \subset L(V)$ corresponding to the decomposition (7), then

$$d_\sigma T(\overline{\chi}_\sigma;\, g_0) = P_\sigma(T(g_0)). \tag{8}$$

That this latter relation may be considered as a generalization of the results on the class operator referred to above will be seen from the following discussion. Recall that the character of the trivial representation is the function $\mathbf{1}$ equal identically to 1 on G, hence the integral (3) corresponds to the class operator for G. By virtue of (b) above it is an intertwining operator for the representation T, i.e. it satisfies

$$T(g)T(\mathbf{1}; g_0)T(g)^{-1} = T(\mathbf{1}; g_0) \qquad \text{for each } g \in G\,.$$

Corollary. Assume (T, V) is an irreducible representation of G on a finite dimensional space V, then

$$T(\mathbf{1}; g_0) = \frac{\chi_T(g_0)}{\dim T} I\,.$$

I denoting the identity operator on the representation space of T, and $\chi_T(g) = \mathrm{Tr}(T(g))$ being the character of T.

In fact, T is proportional to the identity by virtue of the Schur lemma and the coefficient of proportionality can be obtained by evaluating the trace under the integral sign in (3). The fact that the right-hand side is the orthogonal projection of $T(g_0)$ onto the space of scalar operators can also be verified directly by observing that the orthogonal complement of the latter space is the space of operators with vanishing trace. Specialized to the case of $SU(2)$ this equality is precisely the one established in the papers referred to above.

Examining the integral in (3) a bit closer one sees that it has an additional invariance property, namely if $Z(g_0) \subset G$ denotes the centralizer of g_0 in G, i.e. $Z(g_0) = \{h \in G \mid hg_0 = g_0 h\}$, then also

$$T(\rho(h)f; g_0) = T(f; g_0) \qquad \text{for each } h \in Z(g_0)\,. \tag{9}$$

This means that one can replace the given function f by its shift along the co-sets of $Z(g_0)$ without affecting the value of the integral (3). This allows us to use the well known technique of transferring integrals from the group to the homogeneous space $G/Z(g_0)$, thus passing from the map (4) to the map of the space of functions on the co-set space $G/Z(g_0)$. We recall it briefly here.

Note that the space $G/Z(g_0)$ can be naturally identified with the conjugacy class $C(g_0) = \{gg_0g^{-1} \mid g \in G\}$ in G by means of the correspondence $G/Z(g_0) \ni xZ(g_0) \leftrightarrow xg_0x^{-1} \in G/Z(g_0)$. In this way the action of G on the conjugacy class by conjugation corresponds to the usual left action on $G/Z(g_0)$.

In particular, given any function $f \in C(G)$ one can define a continuous function $\tilde{f}$ on the quotient space $G/Z(g_0)$ by averaging over co-sets, that is by setting

$$\tilde{f}(xZ(g_0)) = \int_{Z(g_0)} f(xh)\mathrm{d}h\,. \tag{10}$$

The map $C(G) \ni f \to \tilde{f} \in C(G/Z(g_0))$ is surjective and therefore the map (4) gives rise to a unique map $\tilde{T}: C(G/Z(g_0)) \to L(V)$ by setting

$$\tilde{T}(\varphi;\, g_0) := T(f;\, g_0) \qquad \varphi \in C(G/Z(g_0))$$

where $f \in C(G)$ is any function such that $\tilde{f} = \varphi$.

This procedure can be given a slightly different description as follows. For any $x \in G$ let the corresponding co-set $xZ(g_0) \in G/Z(g_0)$ be denoted by $\dot{x}$ and let $\mathrm{d}\mu(\dot{x})$ be the (unique) invariant measure on $G/Z(g_0)$ defined by the relation

$$\int_G f(x)\mathrm{d}x = \int_{G/Z(g_0)} \mathrm{d}\mu(\dot{x}) \int_{Z(g_0)} f(xh)\mathrm{d}h\,. \tag{11}$$

Now observing that $T(xg_0x^{-1})$ depends only on the co-set $\dot{x} = xZ(g_0)$ of x we can write $T(xg_0x^{-1}) = T(\dot{x})$ and regard the map $x \to T(xg_0x^{-1})$ as the function $\dot{x} \to T(\dot{x})$ on $G/Z(g_0)$. We see that

$$\tilde{T}(\varphi; g_0) = \int_{G/Z(g_0)} \varphi(\dot{x})T(\dot{x})\mathrm{d}\mu(\dot{x}) \qquad \varphi \in C(G/Z(g_0))\,. \tag{12}$$

In view of (5) we then have

$$T(g)T(\varphi; g_0)T(g)^{-1} = T(\lambda(g)\varphi; g_0) \qquad \varphi \in C(G/Z(g_0))$$

showing that the map $C(G/Z(g_0)) \ni \varphi \to T(\varphi; g_0)$ is covariant with respect to the action of G.

3. The case of $SU(2)$

The general scheme given above will be applied in this section to the case of the group $SU(2)$. In particular we shall show how to obtain the explicit expression for the class operator for $SU(2)$, derived in the papers mentioned. Recall that

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \,\middle|\, a, b \in \mathbf{C},\, |a|^2 + |b|^2 = 1 \right\}.$$

We identify the Lie algebra $su\,(2)$ of $SU(2)$ with the space of anti-hermitean traceless 2×2 matrices so that $\{i\sigma_\alpha\}_{\alpha=1}^3$ is a basis of $su\,(2)$ over the reals. Here of course σ_α are the Pauli matrices. In particular the map

$$\mathbf{R}^3 \ni (t_1, t_2, t_3) \mapsto \exp i(t_1\sigma_1 + t_2\sigma_2 + t_3\sigma_3) \in SU(2)$$

is surjective and the set $\Lambda_e = \{2\pi \mathrm{i}(\mathbf{Z}\sigma_1 + \mathbf{Z}\sigma_2 + \mathbf{Z}\sigma_3)\}$, where $\mathbf{Z}$ stands for the set of integers, is mapped on the unit matrix $\mathbf{I} \in SU(2)$.

Each unitary matrix is conjugate to a diagonal one, which is determined up to a permutation of its diagonal elements, hence the conjugacy classes for $SU(2)$ (except the trivial ones) are in a $1 \leftrightarrow 2$ correspondence with elements of the maximal torus $U(1)$ in $SU(2)$, which we choose to be $U(1) = \{\exp \mathrm{i}(\psi/2)\sigma_3 \mid \psi \in \mathbf{R}\}$. Now set $g(\psi) = \exp \mathrm{i}(\psi/2)\sigma_3)$ and recall $gg(\psi)g^{-1} = \exp \mathrm{i}(\psi/2)\mathrm{Ad}(g)\sigma_3)$, where the adjoint representation $g \to \mathrm{Ad}(g)$ taken with respect to the basis $\mathrm{i}\sigma_\alpha\}_{\alpha=1}^3$ is the standard covering $SU(2) \to SO(3)$.

Thus we see that points of the conjugacy class $C(g(\psi))$ of the element $g(\psi)$ are in a one-to-one correspondence with the points of the sphere $S_\psi = \{\mathrm{i}(\psi/2)\boldsymbol{n} \cdot \boldsymbol{\sigma} \mid \boldsymbol{n} \in \mathbf{R}^3;\ |\boldsymbol{n}| = 1\}$ in $su\,(2) \simeq \mathbf{R}^3$ with radius $|\psi/2|$.

We denote

$$(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \boldsymbol{n}(\theta, \phi)$$

so that $[0, \pi] \times [0, 2\pi[\ni (\theta, \phi) \mapsto \exp(\mathrm{i}(\psi/2)\boldsymbol{n}(\theta, \phi) \cdot \boldsymbol{\sigma})$ is a parameterization of $C(g(\psi))$ and the invariant integral on $C(g(\psi))$ defined by (11) is given as

$$\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} f\left(\exp\left(\mathrm{i}\frac{\psi}{2}\boldsymbol{n}(\theta, \phi) \cdot \boldsymbol{\sigma}\right)\right) \sin\theta \mathrm{d}\theta \mathrm{d}\phi.$$

For the case of the character χ_s of the irreducible representation (T_s, V_s) of $SU(2)$ of dimension $2s + 1$ we have, recalling that the characters are central functions, the following expression:

$$\begin{aligned} T_j(\overline{\chi_s}; g(\psi)) &= \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} \overline{\chi_s}(g(\theta)) T_j \exp\left(\mathrm{i}\frac{\psi}{2}\boldsymbol{n}(\theta, \phi) \cdot \boldsymbol{\sigma}\right) \sin\theta \, d\theta d\phi \\ &= \frac{1}{2\pi}\int_0^\pi\int_0^{2\pi} \sin\left((2s+1)\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) T_j \exp\left(i\frac{\psi}{2}\boldsymbol{n}(\theta, \phi) \cdot \boldsymbol{\sigma}\right) \mathrm{d}\theta \mathrm{d}\phi \end{aligned} \tag{13}$$

where we have used the fact that

$$\chi_s(g(\theta)) = \frac{\sin[(2s+1)\theta/2]}{\sin(\theta/2)}.$$

On the other hand, the identification $L(V_j) \simeq V_j \otimes V_j$ allows us to apply the Clebsch–Gordan decomposition

$$V_j \otimes V_j = \bigoplus_{s=0}^{2j} V_s \tag{14}$$

to find projections of a given operator onto subspaces V_j and thus obtain the integrals (13).

However, recall that an identification of $V_j \otimes V_j$ with $L(V_j)$ is given by the map

$$v \otimes w \to T_{v\otimes w} \in L(V_j)$$

where

$$T_{v\otimes w}(x) = (w|x)v \qquad x \in V_j$$

so, as a consequence of the requirement of linearity for this identification we must let $G = SU(2)$ act on $V_j \otimes V_j$ by means of $T_j \otimes \overline{T_j}$, rather then by $T_j \otimes T_j$, $\overline{T_j}$ being the representation conjugate to T_j. This will result in the fact that our CG coefficients are numerically different from the standard ones found in the most sources (e.g. [9–11]).

Thus if $\{e_k\}_{k=-j}^{j}$ is a basis in V_j with respect to which $T_j(g(\psi))$ are diagonal, i.e.

$$T_j(g(\psi))e_k = \mathrm{e}^{\mathrm{i}k\psi} e_k$$

and $\{A_m^s \mid 0 \leqslant s \leqslant 2j,\ -s \leqslant m \leqslant s\}$ is an orthonormal basis of $V_j \otimes V_j$ compatible with the decomposition (14), i.e. such that the vectors $\{A_m^s\}_{m=-s}^{s}$ form a basis for the irreducible subspace V_s, then we introduce the CG coefficients by means of the decomposition

$$e_k \otimes e_l = \sum_{s=0}^{2j} \sum_{m=-s}^{s} c(j, j, s; k, l, m) A_m^s$$

and as usual we normalize the bases so that $c(j, j, s; k, l, m)$ are all real.

Since $c(j, j, s; k, l, m) = 0$, unless $m = k - l$, the above simplifies to

$$e_k \otimes e_l = \sum_{s=0}^{2j} c(j, j, s; k, l, k-l) A_{k-l}^s\,.$$

By diagonality of $T_j(g(\psi))$ we have

$$T_j(g(\psi)) = \sum_{k=-j}^{j} d_k(\psi) e_k \otimes e_k = \sum_{s=0}^{2j} \left(\sum_{k=-j}^{j} c(j, j, s; k, k, 0) d_k(\psi) \right) A_0^s$$

with $d_k(\psi) = \mathrm{e}^{\mathrm{i}k\psi}$ and hence the component of $T_j(g(\psi))$ in V_s is given by the expression

$$P_s T_j(g(\psi)) = \left(\sum_{k=-j}^{j} c(j, j, s; k, k, 0) d_k(\psi) \right) A_0^s \qquad s = 0, 1, \ldots, 2j\,. \tag{15}$$

Thus the projections $P_s T_j(g(\psi))$ are, for all values of $\psi \in [0, 2\pi]$, proportional to the single operator A_0^s, the only dependence on ψ being through the factor of proportionality

$$\delta^s(\psi) = \sum_{k=-j}^{j} c(j, j, s; k, k, 0) d_k(\psi)\,.$$

Note that for each s the operators $A_0^s \in L(V_j)$ satisfy the relation

$$T_j(g(\psi)) A_0^s T_j(g(\psi))^{-1} = A_0^s$$

from which it follows that A_0^s are diagonal with respect to the basis $\{e_k\}_{k=-j}^{j}$ and in particular A_0^0 is, possibly up to a constant, an identity operator.

Now one can express A_0^s in terms of the canonical basis $\{e_k \otimes e_l\}$ again. In fact, by virtue of unitarity of the matrix of (modified) Clebsch–Gordan coefficients (which holds since they are defined as coefficients of the transition matrix between the orthonormal bases $\{e_k \otimes e_l\}$ and $\{A_m^s \mid 0 \leqslant s \leqslant 2j,\ -s \leqslant m \leqslant s\}$) and using also the fact they are real we have

$$A_0^s = \sum_{p=-j}^{j} c(j, j, s; p, p, 0) e_p \otimes e_p \tag{16}$$

and therefore we get

$$P_s T_j(g(\psi)) = \delta^s(\psi) \sum_{p=-j}^{j} c(j, j, s; p, p, 0) e_p \otimes e_p\,.$$

Finally, for the integral in (13) we have

$$\begin{aligned} T_j(\overline{\chi_s}; g(\psi)) &= \frac{1}{2s+1} \sum_{k=-j}^{j} c(j, j, s; k, k, 0) d_k(\psi) A_0^s \\ &= \frac{1}{2s+1} \left(\sum_{k=-j}^{j} c(j, j, s; k, k, 0) d_k(\psi) \right) \sum_{p=-j}^{j} c(j, j, s; p, p, 0) e_p \otimes e_p\,. \end{aligned} \tag{17}$$

The problem of computing explicit vales of the modified CG coefficients which enter (17) is of a quite different nature and will not be treated here. We just confine ourselves to the simplest special case, namely the case $s = 0$, and check that it gives the original value of the class operator.

To this end we observe that (16) implies the coefficients $c(j, j, 0; p, p, 0)$ are independent of p and by orthogonality

$$c(j, j, 0; p, p, 0) = \frac{1}{\sqrt{2j+1}}\,.$$

Thus, setting $s = 0$ in (17) we get

$$T_j(1; g(\psi)) = \frac{1}{2j+1} \sum_{k=-j}^{j} d_k(\psi) \sum_{p=-j}^{j} e_p \otimes e_p = \frac{\sin[(2j+1)\psi/2]}{(2j+1)\sin(\psi/2)}\, I$$

in agreement with the quoted results.

4. Conclusions

We have shown that the construction of the class operator for the $SU(2)$ group as well as for other compact groups can be considerably generalized by employing constructions of the general group representation theory. Considering the problem of decomposition of the representation operator into components transforming according to irreducible representations we have obtained not only an analog of the class operator in the case of arbitrary compact groups, but also another interesting class of operators with a natural interpretation. Our formulation seems to be a good starting point for further generalization to the case of non-compact groups.

Note added. After the submission of the paper, P Kasperkovitz has kindly pointed out to us the papers [12] and [13], where the problem of decomposing the conjugation representation (termed there the *tensor representation*) is studied in detail. In particular, some of the arguments of our section 3 can already be found there, e.g. the definition and several properties of modified CG coefficients, termed there the *coupling coefficients*, are given in [12]. The overall methods and aims of those two papers as compared ours are, however, apparently rather distinct.

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